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A three-parameter solution of the static Einstein–Maxwell equations

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Abstract. The solution represents the external field of an isolated mass carrying electric charge and dipole moment.

1. Introduction

A class of solutions of the axially symmetric electrostatic problem in general relativity was obtained by Weyl (1917). Weyl's class consists of space–times in which the gravitational and electrostatic potentials are functionally related. Another class of solutions, without spatial symmetry but with functionally related potentials, was found by Majumdar (1947) and by Papapetrou (1947). I obtained a solution (Bonnor 1966), without functionally related potentials, referring to a massive electric dipole. This was found by transforming the Kerr vacuum solution into an electrostatic space–time; it depends on two parameters. This procedure has since (Önengüt and Serdaroğlu 1975) been extended to the Tomimatsu–Sato vacuum solutions, which have been transformed to yield some (extremely complicated) two-parameter electrostatic space–times.

Recently Chandrasekhar (1978) has suggested a new way of solving the stationary, axially symmetric vacuum equations, and in this paper I use his method to obtain a new, comparatively simple axially symmetric electrostatic solution depending on three parameters.

2. Field equations

The equations to be solved are those of Einstein–Maxwell theory in the absence of matter:

$$R_{ik} = 2F_i^{\ a}F_{ka} - \tfrac{1}{2}g_{ik}F^{ab}F_{ab} \tag{2.1}$$

where R_{ik} is the Ricci tensor and F_{ik} is the electromagnetic field tensor which satisfies Maxwell's equations,

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{2.2}$$

$$F^{ik}{}_{;k} = 0. \tag{2.3}$$

In the electrostatic problem all variables are independent of the time t ($\equiv x^4$), and (2.2) is satisfied if we take

$$F_{ik} = \kappa_{i;k} - \kappa_{k;i} \tag{2.4}$$

with

$$\kappa_i = \delta_i^4 \phi(x^\nu) \qquad (\nu = 1, 2, 3), \tag{2.5}$$

ϕ being the electrostatic potential.

The metric in the axially symmetric case may be taken as (Weyl 1917)

$$ds^2 = -e^\lambda(du^2 + d\theta^2) - F^{-2}\Delta^2 d\psi^2 + F^2 dt^2 \tag{2.6}$$

where $x^1 \equiv u$, $x^2 \equiv \theta$, $x^3 \equiv \psi$ and λ, Δ, F and ϕ are functions of u and θ only. Because of the field equations (2.1) Δ satisfies

$$\Delta_{11} + \Delta_{22} = 0 \tag{2.7}$$

where subscripts 1 and 2 mean differentiation with respect to u and θ respectively.

The entire solution is determined by two equations, the (44) equation of (2.1) and the equation for $i = 4$ of (2.3). Taking linear combinations of these and putting

$$X = F + \phi \qquad Y = F - \phi \tag{2.8}$$

we obtain two equations equivalent to two given by Chandrasekhar (his (47)):

$$(X + Y)\nabla^2 X = 2\nabla X . \nabla X \tag{2.9}$$

$$(X + Y)\nabla^2 Y = 2\nabla Y . \nabla Y \tag{2.10}$$

where ∇^2 and ∇ denote respectively the Laplacian and gradient operators in the three-space with metric

$$e^\tau(du^2 + d\theta^2) + \Delta^2 d\psi^2, \tag{2.11}$$

$\tau(u, \theta)$ being arbitrary. Once X and Y are found, the function λ in (2.6) is determined up to an additive constant by the other field equations (2.1) (Bonnor 1953). Weyl's solutions are obtained by allowing X and Y (or F and ϕ) to be functionally related.

In this paper we shall choose u and θ to be prolate spheroidal coordinates so that the appropriate solution of (2.7) is

$$\Delta = a \sinh u \sin \theta, \tag{2.12}$$

a being an arbitrary constant. If we demand that the three-space (2.11) is flat we find

$$e^\tau = a^2(\cosh^2 u - \cos^2 \theta),$$

though this will not be used in the sequel. Written out in these coordinates, (2.9) and (2.10) become

$$(X + Y)(X_{11} + X_{22} + X_1 \coth u + X_2 \cot \theta) = 2(X_1^2 + X_2^2) \tag{2.13}$$

$$(X + Y)(Y_{11} + Y_{22} + \coth u Y_1 + \cot \theta Y_2) = 2(Y_1^2 + Y_2^2). \tag{2.14}$$

3. The solution

It is convenient to introduce new independent variables

$$\eta = \cosh\theta \qquad \mu = \cos\theta; \tag{3.1}$$

(2.13) and (2.14) take the form

$$(X+Y)\left((\eta^2-1)\frac{\partial^2 X}{\partial\eta^2}+2\eta\frac{\partial X}{\partial\eta}+(1-\mu^2)\frac{\partial^2 X}{\partial\mu^2}-2\mu\frac{\partial X}{\partial\mu}\right)$$
$$=2\left[(\eta^2-1)\left(\frac{\partial X}{\partial\eta}\right)^2+(1-\mu^2)\left(\frac{\partial X}{\partial\mu}\right)^2\right] \tag{3.2}$$

$$(X+Y)\left((\eta^2-1)\frac{\partial^2 Y}{\partial\eta^2}+2\eta\frac{\partial Y}{\partial\eta}+(1-\mu^2)\frac{\partial^2 Y}{\partial\mu^2}-2\mu\frac{\partial Y}{\partial\mu}\right)$$
$$=2\left[(\eta^2-1)\left(\frac{\partial Y}{\partial\eta}\right)^2+(1-\mu^2)\left(\frac{\partial Y}{\partial\mu}\right)^2\right]. \tag{3.3}$$

Adapting a prescription of Chandrasekhar (1978) we seek solutions of (3.2) and (3.3) of the form

$$X=(a_1+a_2F)/(a_3+a_4F) \qquad Y=(b_1+b_2G)/(b_3+b_4G) \tag{3.4}$$

where $a_1\ldots a_4$, $b_1\ldots b_4$ are real constants and

$$F=f_1\eta+f_2\mu \qquad G=g_1\eta+g_2\mu,$$

f_1, f_2, g_1, g_2 being further real constants.

The substitution of (3.4) into (3.2) and (3.3) is a long but straightforward calculation. The result can be expressed in the form

$$F\equiv\tfrac{1}{2}(X+Y)=K(1-BU^{-1}-CV^{-1}) \tag{3.5}$$

$$\phi\equiv\tfrac{1}{2}(X-Y)=K(CV^{-1}-BU^{-1}) \tag{3.6}$$

where

$$U=B+A\mu-\eta \qquad V=C+A\mu+\eta, \tag{3.7}$$

A, B, C and K being constants satisfying

$$BC=A^2-1. \tag{3.8}$$

In arriving at (3.6) we have taken advantage of the fact that ϕ, the electrostatic potential, occurs in the field equations only in the form of its derivatives: this allows us to add to it a constant which brings ϕ into the convenient form (3.6). An inspection of the metric (2.6) and the definition (2.5) of ϕ shows that the constant K can be removed from the solution by the coordinate transformation

$$t=K^{-1}t' \qquad \psi=K\psi'$$

leaving three independent constants in the solution so far, namely a introduced in (2.12) and two of A, B and C. We henceforth assume K removed, and drop the primes on ψ and t.

It is clear from (3.5) and (3.6) that F and ϕ are functionally independent so this solution is not in the Weyl class.

The function λ can be found from the other field equations (2.1) by quadrature. The result is

$$e^{\lambda} = E(\eta^2-\mu^2)^{-3}U^2V^2(UV-CU-BV)^2, \tag{3.9}$$

E being an arbitrary constant which we shall choose so that the metric (2.6) shall obey the condition of elementary flatness on the rotation axis away from singularities (i.e. so that the circumference of a small circle with centre on the axis shall be 2π times its radius). This requires

$$E = a^2. \tag{3.10}$$

The solution can now be written

$$ds^2 = -a^2U^2V^2\left[\frac{(UV-BV-CU)^2}{(\eta^2-\mu^2)^3}\left(\frac{d\eta^2}{\eta^2-1}+\frac{d\mu^2}{1-\mu^2}\right)+\frac{(\eta^2-1)(1-\mu^2)\,d\psi^2}{(UV-BV-CV)^2}\right]$$
$$+\frac{(UV-BV-CU)^2}{U^2V^2}\,dt^2 \tag{3.11}$$

$$\phi = (CU-BV)U^{-1}V^{-1} \tag{3.12}$$

where η, μ, *defined by* (3.1) *are used instead of* u *and* θ, *where* U *and* V *are given by* (3.7) *and where the constants* A, B, C *and* a *satisfy the relation* (3.8). This is the main result of the paper, but an interpretation follows.

4. Interpretation

The transformation

$$x = a(\eta^2-1)^{1/2}(1-\mu^2)^{1/2}\cos\psi$$
$$y = a(\eta^2-1)^{1/2}(1-\mu^2)^{1/2}\sin\psi$$
$$z = a\eta\mu$$

takes the solution (3.11) and (3.12) into

$$ds^2 = -\left(1+\frac{2(C-B)a}{R}+O(R^{-2})\right)(dx^2+dy^2+dz^2)+\left(1-\frac{2(C-B)a}{R}+O(R^2)\right)dt^2 \tag{4.1}$$

$$\phi = \frac{(B+C)a}{R}-\frac{A(C-B)a^2\mu}{R^2}+\frac{(B^2-C^2)a^2}{R^2}+O(R^{-3}) \tag{4.2}$$

where $R^2 = x^2+y^2+z^2$. The solution is therefore asymptotically flat, the leading terms corresponding to an isolated mass m with electric charge e and electric dipole moment d given by

$$m = (C-B)a \qquad e = (C+B)a \qquad d = a^2A(B-C) = -bm, \tag{4.3}$$

the arbitrary constants now being m, e and b $(=aA)$†.

† For convenience the constant b is taken to refer to an electric dipole moment d defined in (4.3). However, as in classical electrostatics, the dipole term in ϕ can be removed (provided $e \neq 0$) by shifting the origin with a transformation $R^* = R + n\cos\theta$, where n is a suitable constant. It is therefore strictly more correct to regard b as referring to the distribution of higher multipoles.

The solution (3.11) and (3.12) may be written in terms of spherical coordinates r, θ and ψ by means of the transformation

$$a\eta = r - \tfrac{1}{2}m \qquad \mu = \cos\theta. \tag{4.4}$$

The result is

$$ds^2 = -W^2[P^2Q^{-3}(Z^{-1}\,dr^2 + d\theta^2) + ZP^{-2}\sin^2\theta\,d\psi^2] + P^2W^{-2}\,dt^2 \tag{4.5}$$

$$\phi = W^{-1}[e(r-\tfrac{1}{2}m) - mb\cos\theta] \tag{4.6}$$

where

$$\begin{aligned} P &= (r-\tfrac{1}{2}m)^2 - a^2 + b^2\sin^2\theta \\ Q &= (r-\tfrac{1}{2}m)^2 - a^2\cos^2\theta \\ W &= r^2 - (b\cos\theta + \tfrac{1}{2}e)^2 \\ Z &= (r-\tfrac{1}{2}m)^2 - a^2 \end{aligned} \tag{4.7}$$

and the arbitrary constants a, b, e and m are connected by

$$a^2 = b^2 + \tfrac{1}{4}(m^2 - e^2), \tag{4.8}$$

a relationship which follows from (3.8) and (4.3). The singularity structure is complicated, but the singularities are all enclosed inside some finite surface; for if l denotes the greater of $|a| + |\tfrac{1}{2}m|$ and $|b| + |\tfrac{1}{2}e|$, then the singularities are within

$$r = l;$$

therefore the ranges of the variables may be taken as

$$r > l \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \psi \leq 2\pi \qquad -\infty < t < \infty. \tag{4.9}$$

This space–time is, of course, incomplete, but it can be regarded as a possible exterior to some charged matter distribution. It is to be emphasised that there are no singularities on the rotation axis for $r > l$.

The following specialisations of the solution may be of interest.

(i) $e = 0$, $m \neq 0$, $b \neq 0$. The solution reduces (if m is replaced by $2m$ and some unimportant changes in notation are made) to the dipole solution (Bonnor 1966).

(ii) $b = 0$, $m^2 \neq e^2$. It reduces to a (non-spherically symmetric) Weyl solution.

(iii) $b = 0$, $m^2 = e^2$. The solution is equivalent to the Reissner–Nordström solution with $m^2 = e^2$.

(iv) $m = 0$, $e = 0$. The solution is flat.

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